Blacktown Boys' High School

2022

HSC Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided for this paper
- All diagrams are not drawn to scale
- In Questions 11−14, show all relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 3−7)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 8−14)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

STUDENT NAME: ____

Students are advised that this is a trial examination only and cannot in any way guarantee the content or format of the 2022 Higher School Certificate Examination.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1–10. Only the multiple choice answer sheet will be marked.

---

1 Three women and three men are to be seated around a circular table. In how many ways can this be done if the men and women must alternate?

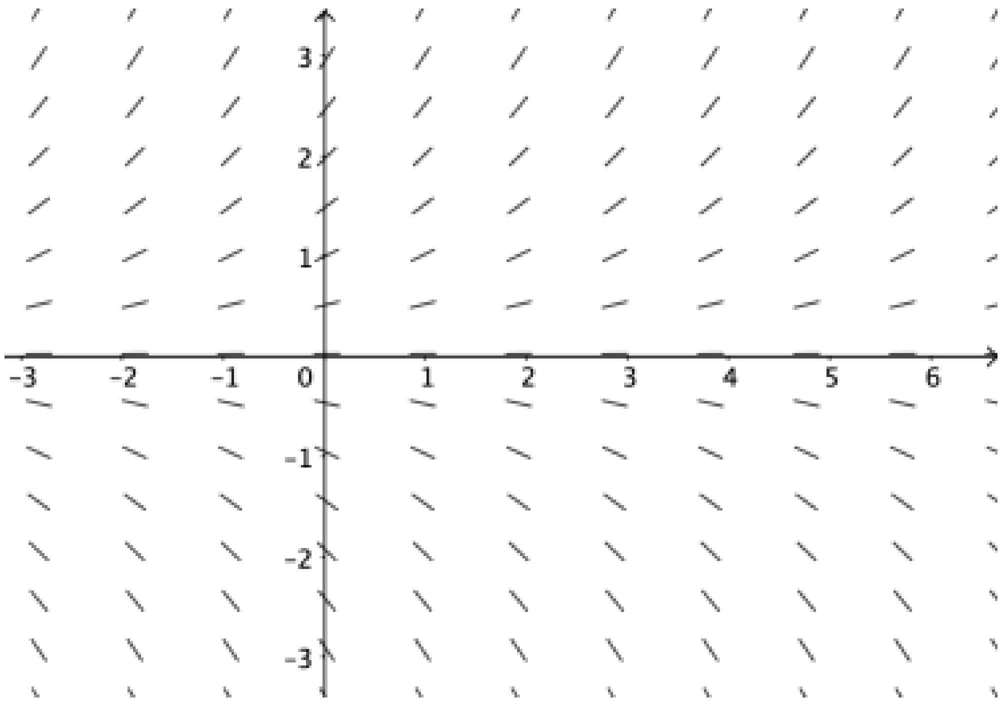
A. $3! \times 2!$

B. $2! \times 2!$

C. $5!$

D. $6!$

2 Consider the slope field below:



What is the possible differential equation for the slope field?

A. $\dfrac{dy}{dx} = -\dfrac{x}{2}$

B. $\dfrac{dy}{dx} = -\dfrac{y}{2}$

C. $\dfrac{dy}{dx} = \dfrac{x}{2}$

D. $\dfrac{dy}{dx} = \dfrac{y}{2}$

Examination continues on the next page

3 A coin is biased such that the probability of a head on any toss is 0.69.
Which expression states the probability of a head appearing twice on this coin if this coin is tossed 50 times?

A. $\binom{50}{2} \times 0.31^2 \times 0.69^{48}$

B. $50 \times 0.31^2 \times 0.69^{48}$

C. $\binom{50}{2} \times 0.69^2 \times 0.31^{48}$

D. $50 \times 0.69^2 \times 0.31^{48}$

4 The polynomial $4x^4 - 2x^3 + 9x - 10$ has zeroes α, β, γ, and δ.
What is the value of $\alpha\beta\gamma\delta(\alpha + \beta + \gamma + \delta)$?

A. $\frac{5}{4}$

B. $-\frac{5}{4}$

C. $\frac{9}{4}$

D. $-\frac{9}{4}$

5 Which expression is equal to $\int \cos^2 4x \, dx$?

A. $\frac{x}{2} + \frac{\sin 8x}{16} + C$

B. $\frac{x}{2} - \frac{\sin 8x}{16} + C$

C. $\frac{x}{2} - \frac{\sin 4x}{16} + C$

D. $\frac{x}{2} + \frac{\sin 4x}{16} + C$

Examination continues on the next page

6 A committee of 3 students is to be chosen from a group consisting of 5 students who are girls and 4 students who are boys. What is the probability that 2 boys will be selected?

A. $\frac{1}{84}$

B. $\frac{5}{84}$

C. $\frac{1}{14}$

D. $\frac{5}{14}$

7 What is the vector projection of $\boldsymbol{p} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ onto $\boldsymbol{q} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$?

A. $\boldsymbol{i} + \boldsymbol{j}$

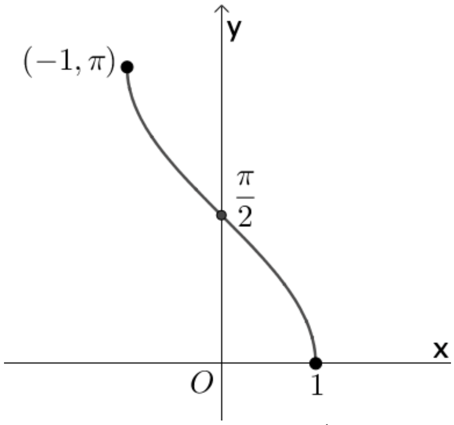
B. $\frac{56}{100}\boldsymbol{i} + \frac{42}{75}\boldsymbol{j}$

C. $\frac{56}{25}\boldsymbol{i} + \frac{42}{25}\boldsymbol{j}$

D. $56\boldsymbol{i} + 42\boldsymbol{j}$

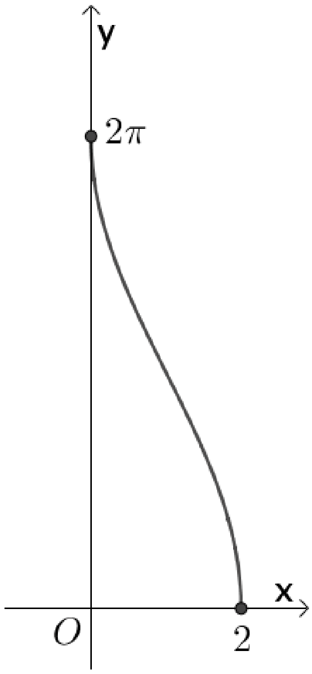
Examination continues on the next page

8 The graph of $y = \cos^{-1} x$ is shown below:

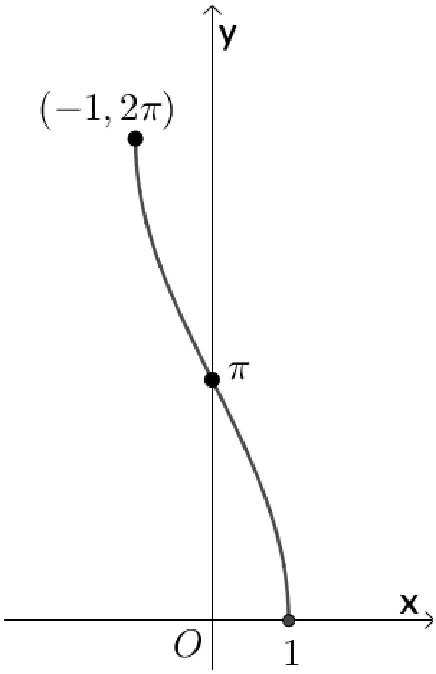


Which of the following is the graph of $y = 2\cos^{-1}(x-1)$?

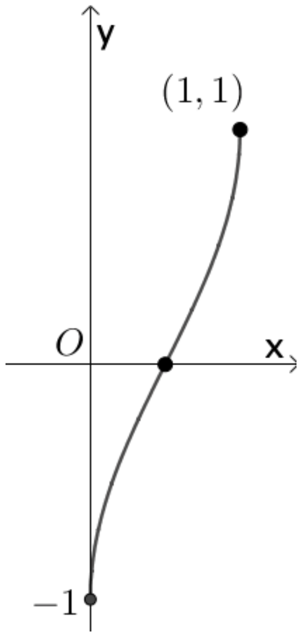
A.



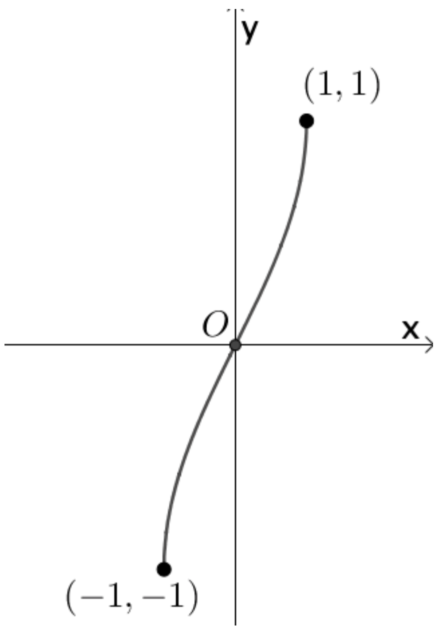
B.



C.



D.



Examination continues on the next page

9 Jacob draws a vector from the origin to the point $A(2,-4)$. Then, he draws another vector $2\boldsymbol{i}+3\boldsymbol{j}$ from the point A, ending in point B. How far is point B from the origin?

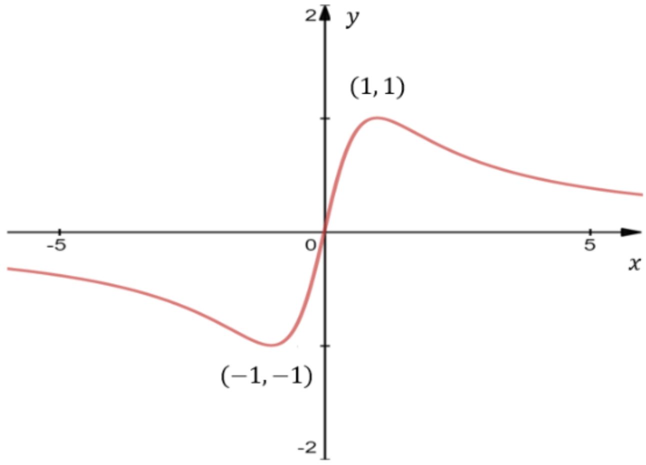
A. $\sqrt{15}$ units

B. $\sqrt{16}$ units

C. $\sqrt{17}$ units

D. $\sqrt{19}$ units

10 The graph of $y=f(x)$ is shown below:



What is the domain and range of $y=f^{-1}(x)$?

A. Domain: all real x
Range: $[-1,1]$

B. Domain: $[-1,1]$
Range: all real y

C. Domain: $[-1,1]$
Range: $[-1,1]$

D. Domain: all real x
Range: all real y

End of Section 1
Examination continues on the next page

Section II

60 Marks
Attempt Questions 11−14
Start each question in a SEPARATE booklet. Extra writing booklets are available.

For Questions 11−14, your responses should all include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

a) i) Differentiate $y = 3x\sin^{-1}(2x)$ **2**

ii) Evaluate $\int_0^{\frac{1}{3}} \frac{-1}{\sqrt{\frac{1}{9} - x^2}}\,dx$ **2**

b) Solve $\frac{-x}{2x+1} \leq \frac{1}{4}$ **3**

c) Given that $X \sim B(n, p)$, $\mu = 3$, and $\sigma^2 = 2$, evaluate:

i) p **2**

ii) n **1**

d) Solve $3\sin 2x = \cos x$, for $0 \leq x \leq 2\pi$. **2**
Round your answers to 2 decimal places where necessary.

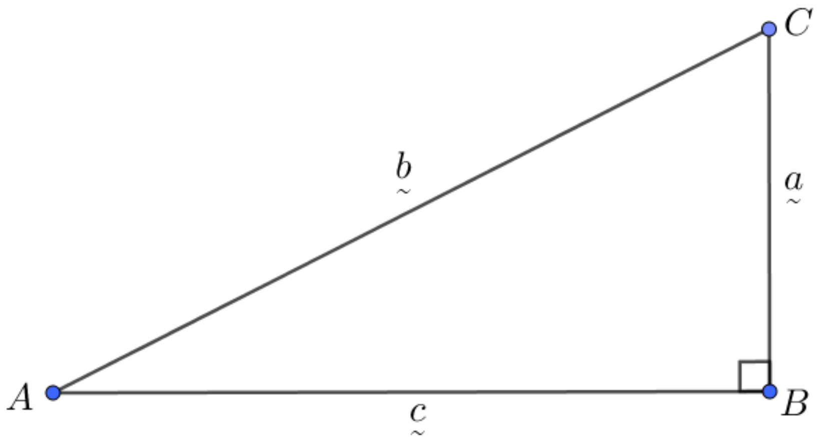
e) Use the substitution $u = x^3$ to evaluate **3**

$$\int_0^1 x^2 e^{x^3}\,dx$$

Examination continues on the next page

Question 12 (15 marks)

a) Consider ΔABC below where $\overrightarrow{AB} = \underset{\sim}{c}$, $\overrightarrow{BC} = \underset{\sim}{a}$, $\overrightarrow{AC} = \underset{\sim}{b}$, and $\angle ABC = 90°$.



Let M be the midpoint of AC.

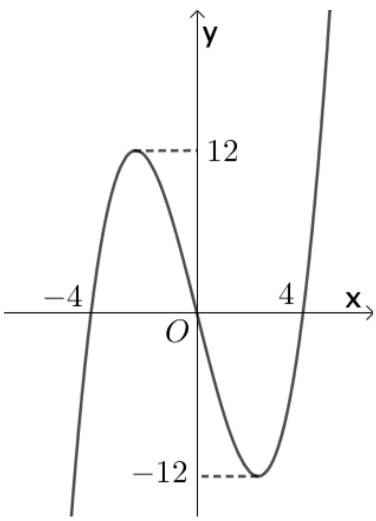
i) Explain why $\overrightarrow{MB} = \underset{\sim}{c} - \frac{\underset{\sim}{b}}{2}$ **1**

ii) Find an expression for $\overrightarrow{MC}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$, and $\underset{\sim}{c}$. **1**

b) i) Express $\sqrt{3}\sin x - \cos x$ in the form $R\sin(x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. **2**

ii) Hence, solve $\sqrt{3}\sin x - \cos x = 1$, for $0 \le x \le 2\pi$. **2**

c) The graph of $y = f(x)$ is shown below:



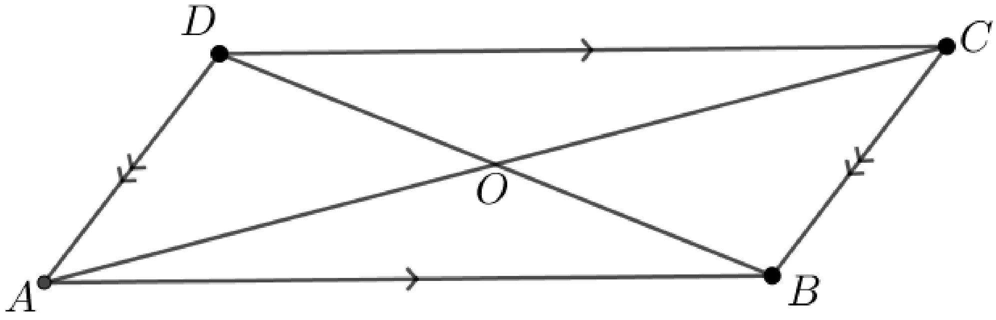
Sketch:

i) $y = \frac{1}{f(x)}$ **2**

ii) $y = \sqrt{f(x)}$ **2**

Question 12 continues on next page

d) Prove, using vectors, that the diagonals of a parallelogram bisect each other. **2**



e) A restaurant knows that 33% of customers will order a take-away meal after dining in the restaurant. **3**
In one particular week, the restaurant took 600 bookings.
Using the normal distribution table below, determine the probability that 200 will order a take-away meal after dining in the restaurant.

	0	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0	0	0.004	0.008	0.012	0.016	0.02	0.024	0.028	0.032	0.036
0.1	0.04	0.044	0.048	0.052	0.056	0.06	0.064	0.068	0.071	0.075
0.2	0.079	0.083	0.087	0.091	0.095	0.099	0.103	0.106	0.11	0.114
0.3	0.118	0.122	0.126	0.129	0.133	0.137	0.141	0.144	0.148	0.152
0.4	0.155	0.159	0.163	0.166	0.17	0.174	0.177	0.181	0.184	0.188
0.5	0.192	0.195	0.199	0.202	0.205	0.209	0.212	0.216	0.219	0.222

Examination continues on the next page

Question 13 (15 marks)

a) A large cylindrical tank is leaking. The volume, V, of water left in the tank at any given time, t, is given by

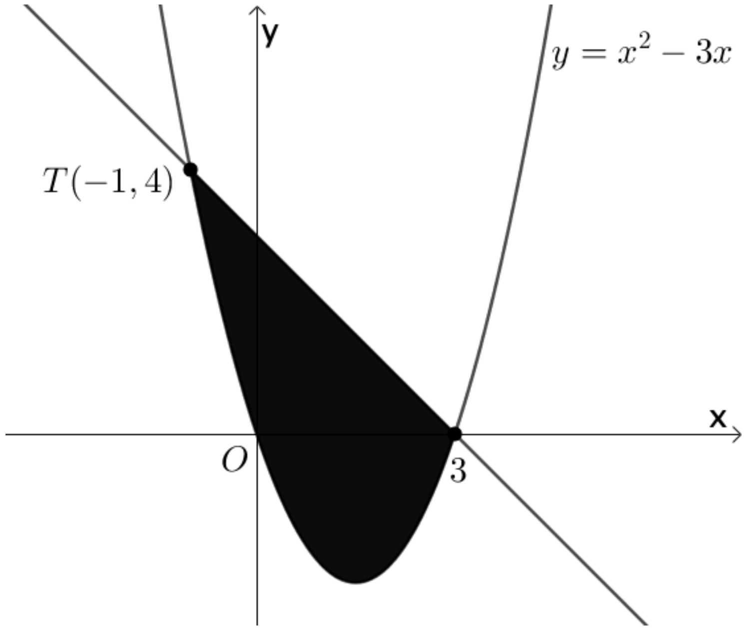
$$\frac{dV}{dt} = -k\sqrt{V}$$

where k is a constant.

i) Find the general solution of the differential equation above. **2**

ii) The tank initially holds 100 litres of water and is leaking at a constant rate of 5L/min. How long will it take for the tank to be empty? **2**

b) Part of the graph of $y = x^2 - 3x$ is shown below. A line is drawn through the point $T(-1, 4)$ such that this line intersects the parabola again at the point $(3, 0)$, as shown below:



i) Show that the equation of the line through $T(-1, 4)$ and $(3, 0)$ is $x + y - 3 = 0$. **1**

ii) The shaded region in the diagram above is rotated about the x-axis. Calculate the exact volume of the solid formed. **2**

c) i) Show that $\dfrac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta} = \tan 3\theta$ **2**

ii) Hence, solve $\dfrac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta} = 1$, for $0 \leq \theta \leq 2\pi$. **1**

d) A coin is tossed 30 times. **2**

Let X be the number of heads. Find the probability that the number of heads is within one standard deviation of the mean using the binomial distribution. Round off your answer to 3 decimal places.

Question 13 continues on the next page

e) i) Show that $\sec(2\sin^{-1}(x))$ can be expressed as $\dfrac{1}{1-2x^2}$ **1**

ii) Hence, or otherwise, solve $\sec(2\sin^{-1}(x)) = -2$. **2**

Examination continues on the next page

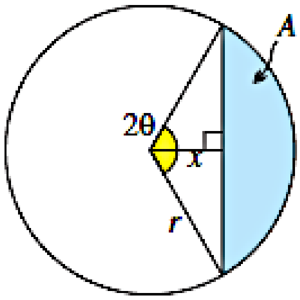
Question 14 (15 marks)

a) i) Prove that $n + 1$ is a factor of $P(n) = 4n^3 + 18n^2 + 23n + 9$. **1**

ii) Hence, use mathematical induction to prove that for all integers $n \geq 1$, **3**

$$1 \times 3 + 3 \times 5 + 5 \times 7 + \cdots + (2n + 1)(2n - 1) = \frac{n(4n^2 + 6n - 1)}{3}$$

b) The diagram below shows a chord of length x from the centre of the circle.



The radius of the circle has length r and the chord subtends an angle of 2θ at the centre of the circle.

i) Show that the shaded area is $A = r^2(\theta - \sin\theta\cos\theta)$. **2**

ii) Explain why $\dfrac{dA}{dt} = \dfrac{dA}{d\theta} \times \dfrac{d\theta}{dx} \times \dfrac{dx}{dt}$ **1**

iii) If the radius is 2 units, how quickly is the shaded area, A, changing if $\dfrac{dx}{dt} = \sqrt{3}$ when $x = 1$? **3**

Question 14 continues on the next page

c) Marine biologists decided to investigate the amount of algae in a pond.

On 1st January 2022, the population of algae in the pond was 1000.
One marine biologist suggested the following mathematical model:

$$\frac{da}{dt} = -7.6 + \frac{1}{10}a(40 - a)$$

where a is the population of algae in the pond after time t years.

i) Show that the differential equation above can be expressed as **1**

$$\frac{da}{dt} = -\frac{1}{10}(a - 2)(a - 38)$$

ii) Given that **3**

$$\frac{-1}{(a-2)(a-38)} = \frac{1}{36(a-2)} - \frac{1}{36(a-38)}$$ (Do NOT prove this.)

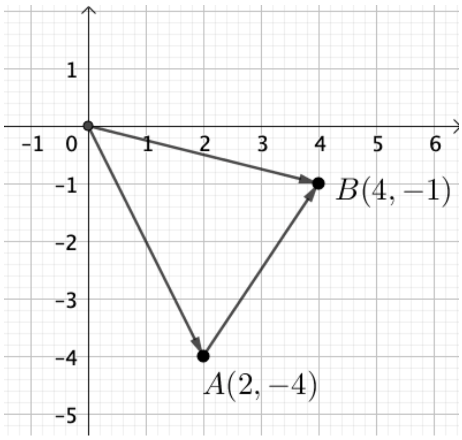
Using this result, show that $t = \frac{5}{18}\log_e\left|\frac{481(a-2)}{499(a-38)}\right|$

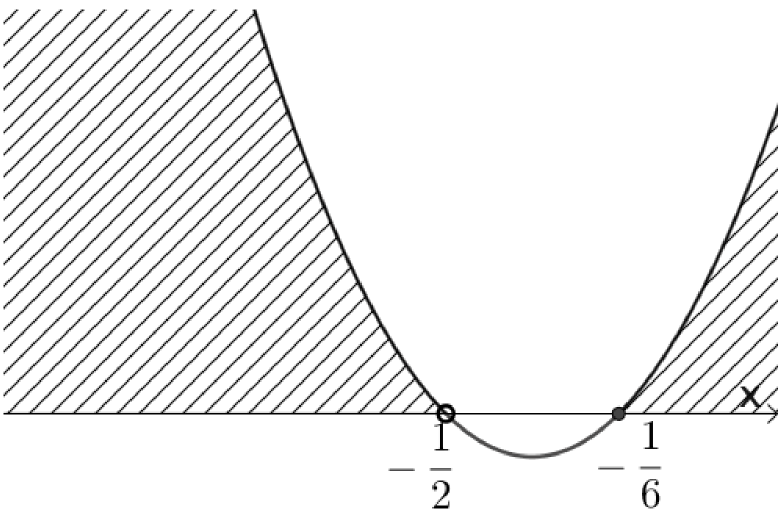
iii) In how many months will the pond be empty of algae? **1**

End of Examination

2022 Year 12 Mathematics Extension 1 Trial
Sample Solutions and Marking Criteria

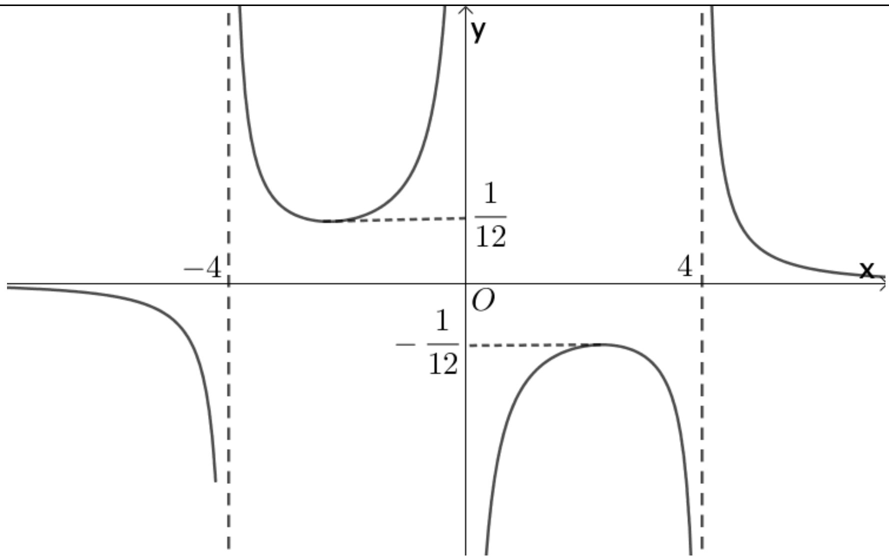
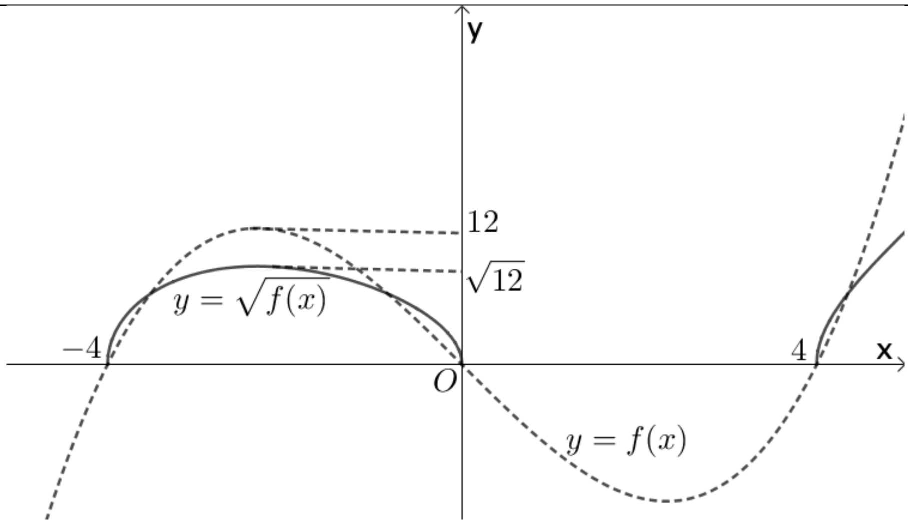
Section 1		
Q1	A One man/woman is fixed in any one seat. ∴ 2 men/women are remaining (can be seated in 2! ways). Remaining people can be seated in 3! Ways ∴ Total number of ways = 3! × 2!	1 Mark Correct Answer
Q2	D When $y > 0, \frac{dy}{dx} > 0$ Only option D satisfies this.	1 Mark Correct Answer
Q3	C $n = 50$ $p = 0.69$ $q = 0.31$ $P\,(2\ heads) = \binom{50}{2} \times 0.69^2 \times 0.31^{48}$	1 Mark Correct Answer
Q4	B $\alpha + \beta + \gamma + \delta = -\frac{-2}{4} = \frac{1}{2}$ $\alpha\beta\gamma\delta = -\frac{10}{4} = -\frac{5}{2}$ $\therefore \alpha\beta\gamma\delta(\alpha + \beta + \gamma + \delta) = -\frac{5}{2} \times \frac{1}{2} = -\frac{5}{4}$	1 Mark Correct Answer
Q5	A $\int \cos^2 4x\, dx = \frac{1}{2}\int (1 + \cos 8x)\, dx$ $\int \cos^2 4x\, dx = \frac{1}{2}\left(x + \frac{\sin 8x}{2}\right) + C$ $\int \cos^2 4x\, dx = \frac{x}{2} + \frac{\sin 8x}{16} + C$	1 Mark Correct Answer
Q6	D $P(2\ boys) = \frac{\binom{4}{2} \times \binom{5}{1}}{\binom{9}{3}} = \frac{5}{14}$	1 Mark Correct Answer

Q7	C $proj_{\cdot q}\,\boldsymbol{p} = \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{\boldsymbol{q}\cdot\boldsymbol{q}} \times \boldsymbol{q}$ $proj_{\cdot q}\,\boldsymbol{p} = \frac{5\times 4 + (-2)\times 3}{4^2+3^2} \times (4\boldsymbol{i}+3\boldsymbol{j})$ $proj_{\cdot q}\,\boldsymbol{p} = \frac{14}{25} \times (4\boldsymbol{i}+3\boldsymbol{j})$ $proj_{\cdot q}\,\boldsymbol{p} = \frac{56}{25}\boldsymbol{i} + \frac{42}{25}\boldsymbol{j}$	1 Mark Correct Answer
Q8	A Option B is the graph of $y = 2\cos^{-1} x$. Option C is the graph of $y = \sin^{-1}(x-1)$. Option D is the graph of $y = \sin^{-1} x$.	1 Mark Correct Answer
Q9	C  $\lvert\overrightarrow{OB}\rvert = \sqrt{4^2+(-1)^2}$ $\lvert\overrightarrow{OB}\rvert = \sqrt{16+1}$ $\lvert\overrightarrow{OB}\rvert = \sqrt{17}$	1 Mark Correct Answer
Q10	B For $y = f(x)$, the domain is all real x and the range is $[-1, 1]$. For $y = f^{-1}(x)$, the domain and range of $y = f(x)$ interchange. So, for $y = f^{-1}(x)$, the domain is $[-1, 1]$ and the range is all real y.	1 Mark Correct Answer

Section 2		
Q11a(i)	$y = 3x\sin^{-1}(2x)$ $\frac{dy}{dx} = 3x\ \frac{2}{\sqrt{1-4x^2}} + 3\sin^{-1}(2x)$ $\frac{dy}{dx} = \frac{6x}{\sqrt{1-4x^2}} + 3\sin^{-1}(2x)$	2 Marks Correct solution 1 Mark Demonstrates that $\frac{d}{dx}(\sin^{-1}(2x)) = \frac{2}{\sqrt{1-4x^2}}$
Q11a(ii)	$\int_0^{\frac{1}{3}} \frac{-1}{\sqrt{1-9x^2}}\,dx = -\int_0^{\frac{1}{3}} \frac{1}{\sqrt{\frac{1}{9}-x^2}}\,dx$ $\int_0^{\frac{1}{3}} \frac{-1}{\sqrt{1-9x^2}}\,dx = -[\sin^{-1}3x]_0^{\frac{1}{3}}$ $\int_0^{\frac{1}{3}} \frac{-1}{\sqrt{1-9x^2}}\,dx = -(\sin^{-1}1 - \sin^{-1}0)$ $\int_0^{\frac{1}{3}} \frac{-1}{\sqrt{1-9x^2}}\,dx = -\frac{\pi}{2}$	2 Marks Correct solution 1 Mark Demonstrates that $\int_0^{\frac{1}{3}} \frac{-1}{\sqrt{1-9x^2}}\,dx = -[\sin^{-1}3x]_0^{\frac{1}{3}}$
Q11b	$-\frac{x}{2x+1} \le \frac{1}{4}$ $x \ne -\frac{1}{2}$ $4 \times (2x+1)^2 \times \frac{-x}{2x+1} \le \frac{1}{4} \times (2x+1)^2 \times 4$ $-4x(2x+1) \le (2x+1)^2$ $(2x+1)^2 + 4x(2x+1) \ge 0$ $(2x+1)(2x+1+4x) \ge 0$ $(2x+1)(6x+1) \ge 0$  Solution: $x < -\frac{1}{2}, x \ge -\frac{1}{6}$	3 Marks Correct solution 2 Marks Demonstrates that $x \ne -\frac{1}{2}$ AND $(2x+1)(6x+1) \ge 0$ 1 Mark Demonstrates that $x \ne -\frac{1}{2}$ AND Multiplies both sides by $(2x+1)^2$

Q11c(i)	$\mu = np$ $3 = np$ ... (1) $\sigma^2 = npq$ $2 = npq$ ... (2) Sub. (1) into (2) $2 = 3q$ $q = \frac{2}{3}$ $p = 1 - \frac{2}{3} = \frac{1}{3}$	2 Marks Correct solution 1 Mark Demonstrates either $3 = np$ and $2 = npq$
Q11c(ii)	$3 = np$ Sub. $p = \frac{1}{3}$ $3 = n \times \frac{1}{3}$ $n = 9$	1 Mark Correct answer
Q11d	$3\sin 2x = \cos x, 0 \leq x \leq 2\pi$ $3(2\sin x \cos x) = \cos x$ $6\sin x \cos x = \cos x$ $6\sin x \cos x - \cos x = 0$ $\cos x\,(6\sin x - 1) = 0$ $\cos x = 0$ or $\sin x = \frac{1}{6}$ $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ or 0.17 or 2.97 (to 2 d.p.)	2 Marks Correct solution 1 Mark Demonstrates that $3\sin 2x = \cos x$ can be expressed as $6\sin x \cos x = \cos x$ and finds one set of correct solutions
Q11e	Let $u = x^3$ $\frac{du}{dx} = 3x^2$ $du = 3x^2 dx$ $x^2 dx = \frac{du}{3}$ When $x = 1, u = 1^3 = 1$ When $x = 0, u = 0^3 = 0$ $\int_0^1 x^2 e^{x^3} dx = \frac{1}{3}\int_0^1 e^u\, du$ $\int_0^1 x^2 e^{x^3} dx = \frac{1}{3}[e^u]_0^1$ $\int_0^1 x^2 e^{x^3} dx = \frac{1}{3}(e^1 - e^0)$	3 Marks Correct solution 2 Marks Correct integration in terms of u 1 Mark Demonstrates either $x^2 dx = \frac{du}{3}$ OR When $x = 1, u = 1^3 = 1$ When $x = 0, u = 0^3 = 0$

	$\int_0^1 x^2 e^{x^3} dx = \frac{1}{3}(e-1)$	
Q12a(i)	We know that $\overrightarrow{AB} = \underset{\sim}{c}$. Since M is the midpoint of $\overrightarrow{AC}$, then $\overrightarrow{AM} = \frac{\underset{\sim}{b}}{2}$ $\overrightarrow{MA} = -\frac{\underset{\sim}{b}}{2}$ $\overrightarrow{MB} = \overrightarrow{MA} + \overrightarrow{AB}$ $\overrightarrow{MB} = -\frac{\underset{\sim}{b}}{2} + \underset{\sim}{c}$ $\overrightarrow{MB} = \underset{\sim}{c} - \frac{\underset{\sim}{b}}{2}$, as required	1 Mark Correct explanation
Q12a(ii)	$\overrightarrow{MC} = \overrightarrow{MB} + \overrightarrow{BC}$ $\overrightarrow{MC} = \underset{\sim}{c} - \frac{\underset{\sim}{b}}{2} + \underset{\sim}{a}$	1 Mark Correct answer
Q12b(i)	$\sqrt{3}\sin x - \cos x = R\sin(x-\alpha)$ $\sqrt{3}\sin x - \cos x = R\sin x\cos\alpha - R\cos x\sin\alpha$ Equate coefficients of $\sin x$ and $\cos x$ $R\cos\alpha = \sqrt{3}$ ... (1) $R\sin\alpha = 1$ ... (2) (2) ÷ (1) $\tan\alpha = \frac{1}{\sqrt{3}}$ $\alpha = \frac{\pi}{6}$ $(2)^2 + (1)^2 = 1$ $R^2\sin^2\alpha + R^2\cos^2\alpha = 1^2 + \sqrt{3}^2$ $R^2(\sin^2\alpha + \cos^2\alpha) = 4$ $R^2 = 4$ $R = 2$ $\sqrt{3}\sin x - \cos x = 2\sin\left(x - \frac{\pi}{6}\right)$	2 Marks Correct solution 1 Mark Evaluates either $\alpha = \frac{\pi}{6}$ OR $R = 2$
Q12b(ii)	$\sqrt{3}\sin x - \cos x = 1, 0 \le x \le 2\pi$ $2\sin\left(x - \frac{\pi}{6}\right) = 1$ $\sin\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$	2 Marks Correct solution

	$x - \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}$ $x = \frac{\pi}{3}, \pi$	1 Mark Finds one correct answer of x
Q12c(i)		2 Marks Correct sketch, including the labelling of all significant points 1 Mark Some significant points labelled
Q12c(ii)		2 Marks Correct sketch, including correct curvature of $y = \sqrt{f(x)}$ relative to $y = f(x)$ 1 Mark Some significant points labelled
Q12d	Let A be the origin. $\overrightarrow{AB} = \underset{\sim}{a}$ $\overrightarrow{AD} = \underset{\sim}{b}$ AC and BD intersect at O. We have to prove that O is the midpoint of AC and BD. Let $\overrightarrow{AO} = x\overrightarrow{AC}$ and $\overrightarrow{OB} = y\overrightarrow{DB}$ ... (1) Now, $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BD}$ $\overrightarrow{AC} = \underset{\sim}{a} + \underset{\sim}{b}$ Also, $\overrightarrow{DB} = \overrightarrow{DA} + \overrightarrow{AB}$ $\overrightarrow{DB} = -\overrightarrow{AD} + \overrightarrow{AB}$	2 Marks Correct proof 1 Mark Establishes basis to prove that O is the midpoint of AC and BD.

	$\overrightarrow{DB} = -\underset{\sim}{b} + \underset{\sim}{a}$ $\overrightarrow{DB} = \underset{\sim}{a} - \underset{\sim}{b}$ From (1), $\overrightarrow{AO} = x\overrightarrow{AC} = x(\underset{\sim}{a} + \underset{\sim}{b})$ and $\overrightarrow{OB} = y\overrightarrow{DB} = y(\underset{\sim}{a} - \underset{\sim}{b})$ Now, $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$ $\underset{\sim}{a} = x\left(\underset{\sim}{a} + \underset{\sim}{b}\right) + y(\underset{\sim}{a} - \underset{\sim}{b})$ $\underset{\sim}{a} = x\underset{\sim}{a} + x\underset{\sim}{b} + y\underset{\sim}{a} - y\underset{\sim}{b}$ $\underset{\sim}{a} = (x+y)\underset{\sim}{a} + (x-y)\underset{\sim}{b}$ Equate coefficients of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ $x + y = 1$ $x - y = 0$ By inspection, $x = y = \frac{1}{2}$ $\overrightarrow{AO} = \frac{1}{2}\overrightarrow{AC}$ $\overrightarrow{OB} = \frac{1}{2}\overrightarrow{DB}$ Hence, the diagonals of a parallelogram bisect each other, as required.	
Q12e	$p = 0.33$ $q = 0.67$ $\mu = np = 600 \times 0.33 = 198$ $\sigma^2 = npq = 600 \times 0.33 \times 0.67 = 132.66$ $\sigma = \sqrt{132.66} = 11.5178\ldots$ $z = \frac{x-\mu}{\sigma} = \frac{200-198}{11.5178\ldots} = 0.173644\ldots$ Using the table, this equates to 0.068 Probability is 6.8%	3 Marks Correct solution 2 Marks Finds the correct z score 1 Mark Evaluates either $\mu = 198$ or $\sigma = \sqrt{132.66}$
Q13a(i)	$\frac{dV}{dt} = -k\sqrt{V}$ $\frac{1}{\sqrt{V}}dV = -kdt$ $V^{-\frac{1}{2}}dV = -kdt$	2 Marks Correct solution 1 Mark Demonstrates that $\frac{dV}{dt} = -k\sqrt{V}$

	$\int V^{-\frac{1}{2}}\,dV = -k\int dt$ $2V^{\frac{1}{2}} = -kt + C$ $2\sqrt{V} = C - kt$ $\sqrt{V} = C - \frac{k}{2}t$ $V = \left(C - \frac{k}{2}t\right)^2$	derives to $\frac{1}{\sqrt{V}}dV = -kdt$
Q13a(ii)	$V = \left(C - \frac{k}{2}t\right)^2$ When $t = 0, V = 100$ $100 = \left(C - \frac{k}{2}\times 0\right)^2$ $100 = C^2$ $C = \pm 10$ Our equations are $V = \left(10 - \frac{k}{2}t\right)^2$ and $V = \left(-10 - \frac{k}{2}t\right)^2$ When $\frac{dV}{dt} = -5, V = 100$ $-5 = -k\sqrt{100}$ $k = \frac{1}{2}$ Our equations are $V = \left(10 - \frac{1}{4}t\right)^2$ and $V = \left(-10 - \frac{k}{4}t\right)^2$ Sub $V = 0$ into both equations above $0 = \left(10 - \frac{1}{4}t\right)^2$ and $0 = \left(-10 - \frac{k}{4}t\right)^2$ Solving the equation gives $t = \pm 40$ But $t > 0$ only. It would take 40 minutes.	2 Marks Correct solution 1 Mark Derives equation for V i.e. $V = \left(10 - \frac{1}{4}t\right)^2$
Q13b(i)	$m = \frac{4-0}{-1-3} = \frac{4}{-4} = -1$ The equation of the line $y - y_1 = m(x - x_1)$ $y - 0 = -(x - 3)$ $y = -x + 3$ $x + y - 3 = 0$, as required	1 Mark Correct proof demonstrating all steps logically

Q13b(ii)	$V = \pi \int_a^b y^2\, dx$ $V = \pi \int_{-1}^{3} ((3-x)^2 - (x^2-3x)^2)\, dx$ $V = \pi \int_{-1}^{3} (9 + x^2 - 6x - x^4 + 6x^3 - 9x^2)\, dx$ $V = \pi \int_{-1}^{3} (-x^4 + 6x^3 - 8x^2 - 6x + 9)\, dx$ $V = \pi \left[-\frac{x^5}{5} + \frac{3x^4}{2} - \frac{8x^3}{3} - 3x^2 + 9x \right]_{-1}^{3}$ $V = \pi \left(\frac{9}{10} + \frac{229}{30} \right)$ $V = \frac{128\pi}{15}\ u^3$	2 Marks Correct solution 1 Mark Demonstrates the volume of the solid of revolution is given by $V = \pi \int_{-1}^{3} ((3-x)^2 - (x^2-3x)^2)\, dx$
Q13c(i)	$LHS = \frac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta}$ $LHS = \frac{(\sin\theta + \sin 5\theta) + \sin 3\theta}{(\cos\theta + \cos 5\theta) + \cos 3\theta}$ $LHS = \frac{2\sin\left(\frac{5\theta+\theta}{2}\right)\cos\left(\frac{5\theta-\theta}{2}\right) + \sin 3\theta}{2\cos\left(\frac{5\theta+\theta}{2}\right)\cos\left(\frac{5\theta-\theta}{2}\right) + \cos 3\theta}$ $LHS = \frac{2\sin 3\theta\cos 2\theta + \sin 3\theta}{2\cos 3\theta\cos 2\theta + \cos 3\theta}$ $LHS = \frac{\sin 3\theta\,(2\cos 2\theta + 1)}{\cos 3\theta\,(2\cos 2\theta + 1)}$ $LHS = \frac{\sin 3\theta}{\cos 3\theta}$ $LHS = \tan 3\theta$ $LHS = RHS$	2 Marks Correct proof 1 Mark Uses sums to products formula to demonstrate that $LHS = \frac{2\sin 3\theta\cos 2\theta + \sin 3\theta}{2\cos 3\theta\cos 2\theta + \cos 3\theta}$
Q13c(ii)	$\frac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta} = 1, 0 \le \theta \le 2\pi$ $\tan 3\theta = 1$ $3\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}, \frac{21\pi}{4}$ $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}$	1 Mark All answers correct
Q13d	$\mu = np = 30 \times \frac{1}{2} = 15$ $\sigma^2 = npq = 30 \times \frac{1}{2} \times \frac{1}{2} = 7.5$	2 Marks Correct solution

	$\sigma = \sqrt{7.5}$ $P(\mu - \sigma \le X \le \mu + \sigma) = P(12.261 \le X \le 17.739)$ $P(\mu - \sigma \le X \le \mu + \sigma) = P(X = 13, 14, 15, 16, 17)$ $P(\mu - \sigma \le X \le \mu + \sigma)$ $= \binom{30}{13}\left(\frac{1}{2}\right)^{30} + \binom{30}{14}\left(\frac{1}{2}\right)^{30} + \binom{30}{15}\left(\frac{1}{2}\right)^{30} + \binom{30}{16}\left(\frac{1}{2}\right)^{30}$ $+ \binom{30}{17}\left(\frac{1}{2}\right)^{30}$ $(\mu - \sigma \le X \le \mu + \sigma) \approx 0.638$	1 Mark Demonstrates that $P(\mu - \sigma \le X \le \mu + \sigma)$ $= P(12.261 \le X$ $\le 17.739)$
Q13e(i)	Let $\alpha = \sin^{-1} x$ $\sin\alpha = x$ $\sec(2\sin^{-1}(x)) = \sec 2\alpha$ $\sec(2\sin^{-1}(x)) = \frac{1}{\cos 2\alpha}$ $\sec(2\sin^{-1}(x)) = \frac{1}{1 - 2\sin^2\alpha}$ $\sec(2\sin^{-1}(x)) = \frac{1}{1 - 2x^2}$, as required	1 Mark Correct proof demonstrating all steps logically
Q13e(ii)	$\sec(2\sin^{-1}(x)) = -2$ $\frac{1}{1 - 2x^2} = -2$ $1 - 2x^2 = -\frac{1}{2}$ $2x^2 = \frac{3}{2}$ $x^2 = \frac{3}{4}$ $x = \pm\frac{\sqrt{3}}{2}$	2 Marks Correct solution 1 Mark Demonstrates that $\frac{1}{1 - 2x^2} = -2$
Q14a(i)	$P(n) = 4n^3 + 18n^2 + 23n + 9$ $P(-1) = 4 \times (-1)^3 + 18 \times (-1)^2 + 23 \times (-1) + 9$ $P(-1) = -4 + 18 - 23 + 9$ $P(-1) = 0$ $n + 1$ is a factor of $P(n)$	1 Mark Correct proof demonstrating all steps logically
Q14a(ii)	When $n = 1$, $LHS = 1 \times 3 = 3$ $RHS = \frac{1(4 \times 1^2 + 6 \times 1 - 1)}{3} = 3 = LHS$	3 Marks Correct solution

The statement is true for $n = 1$.

Assume the statement is true for $n = k$, where k is an integer $k \geq 1$

$$1 \times 3 + 3 \times 5 + 5 \times 7 + \cdots + (2k+1)(2k-1) = \frac{k(4k^2 + 6k - 1)}{3}$$

Prove the statement is true for $n = k + 1$

$$1 \times 3 + 3 \times 5 + 5 \times 7 + \cdots + (2k+1)(2k-1) + (2k+3)(2k+1) = \frac{(k+1)(4k^2 + 14k + 9)}{3}$$

Proof:

$$LHS = 1 \times 3 + 3 \times 5 + 5 \times 7 + \cdots + (2k+1)(2k-1) + (2k+3)(2k+1)$$

$$LHS = \frac{k(4k^2 + 6k - 1)}{3} + (2k+3)(2k+1)$$

$$LHS = \frac{k(4k^2 + 6k - 1) + 3(2k+3)(2k+1)}{3}$$

$$LHS = \frac{4k^3 + 6k^2 - k + 3(4k^2 + 8k + 3)}{3}$$

$$LHS = \frac{4k^3 + 6k^2 - k + 12k^2 + 24k + 9}{3}$$

$$LHS = \frac{4k^3 + 18k^2 + 23k + 9}{3}$$

Let $P(k) = 4k^3 + 18k^2 + 23k + 9$

From part (i), $k + 1$ is a factor of $P(k)$

Now,

$$\begin{array}{r|rrrr}
 & 4k^2 & +14k & +9 & \\
\hline
k+1 & 4k^3 & +18k^2 & +23k & +9 \\
- & 4k^3 & +4k^2 & & \\
\hline
 & & 14k^2 & +23k & +9 \\
- & & 14k^2 & +14k & \\
\hline
 & & & 9k & +9 \\
- & & & 9k & +9 \\
\hline
 & & & & 0
\end{array}$$

$$LHS = \frac{4k^3 + 18k^2 + 23k + 9}{3}$$

2 Marks

Proves the statement is true for $n = 1$ and demonstrates that for $n = k + 1$,

$$LHS = \frac{4k^3 + 18k^2 + 23k + 9}{3}$$

1 Mark

Proves the statement is true for $n = 1$

<table>
<tr><td></td><td>

$LHS = \frac{(k+1)(4k^2+14k+9)}{3}$

$LHS = RHS$

Hence, by mathematical induction, the statement is true for all integers $n \geq 1$.

</td><td></td></tr>
<tr><td>Q14b(i)</td><td>

$A_{shaded} = A_{sector} - A_{triangle}$

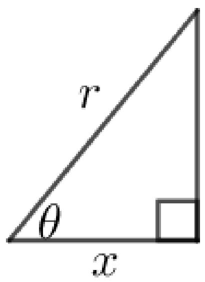
$A_{sector} = \frac{1}{2}r^2\theta$

$A_{sector} = \frac{1}{2}r^2 \times 2\theta$

$A_{sector} = r^2\theta$

$A_{triangle} = \frac{1}{2} \times \text{base} \times \text{height}$

$\text{height} = x$



Using Pythagoras' theorem in the triangle above,

$\text{height of triangle} = \sqrt{r^2 - x^2}$

So, $\text{base} = 2 \times \sqrt{r^2 - x^2}$

Now,

$\sin\theta = \frac{\sqrt{r^2 - x^2}}{r}$

$\therefore \sqrt{r^2 - x^2} = r\sin\theta$

$\therefore \text{base} = 2r\sin\theta$

Also,

$\cos\theta = \frac{x}{r}$

$x = r\cos\theta$

$\therefore \text{height} = r\cos\theta$

So,

$A_{triangle} = \frac{1}{2} \times 2r\sin\theta \times r\cos\theta$

</td><td>

2 Marks
Correct proof demonstrating all steps logically

1 Mark
Demonstrates either

$A_{sector} = \theta r^2$

OR

$A_{triangle} = r^2\sin\theta\cos\theta$

</td></tr>
</table>

	$A_{triangle} = r^2 \sin\theta\cos\theta$ Hence, $A_{shaded} = \theta r^2 - r^2 \sin\theta\cos\theta$ $A_{shaded} = r^2(\theta - \sin\theta\cos\theta)$	
Q14b(ii)	$\frac{dA}{dt}$ is the change in area over time. The shaded area, A, is dependent on the value of θ; therefore, the change in the shaded area, A, changes with the value of θ, i.e., $\frac{dA}{d\theta}$. As θ changes, so does the value of x; therefore, the change in the value of θ changes with the value of x, i.e., $\frac{d\theta}{dx}$. All of these changes happen over time, i.e., $\frac{dx}{dt}$. As a result, the change in the shaded area, A, is dependent on the changes of the values of θ, x, and t, i.e., $\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dx} \times \frac{dx}{dt}$.	1 Mark Correct explanation
Q14b(iii)	$\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dx} \times \frac{dx}{dt}$ $\frac{dx}{dt} = \sqrt{3}$ When $x = 1$, $\theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ Now, $A = r^2(\theta - \sin\theta\cos\theta)$ $A = r^2\theta - r^2\sin\theta\cos\theta$ $\frac{dA}{d\theta} = r^2 - (-r^2\sin^2\theta + r^2\cos^2\theta)$ $\frac{dA}{d\theta} = r^2 - (r^2\cos^2\theta - r^2\sin^2\theta)$ $\frac{dA}{d\theta} = r^2 - \left(r^2(\cos^2\theta - \sin^2\theta)\right)$ Sub $r = 2$ and $\theta = \frac{\pi}{3}$ $\frac{dA}{d\theta} = 2^2 - \left(2^2\left(\cos^2\left(\frac{\pi}{3}\right) - \sin^2\left(\frac{\pi}{3}\right)\right)\right)$ $\frac{dA}{d\theta} = 4 - \left(4\left(\frac{1}{4} - \frac{3}{4}\right)\right)$ $\frac{dA}{d\theta} = 4 - (1 - 3)$ $\frac{dA}{d\theta} = 4 + 2 = 6$ Now,	3 Marks Correct solution 2 Marks Evaluates $\frac{dA}{d\theta} = 6$ AND $\frac{d\theta}{dx} = -\frac{1}{\sqrt{3}}$ 1 Mark Evaluates either $\frac{dA}{d\theta} = 6$ OR $\frac{d\theta}{dx} = -\frac{1}{\sqrt{3}}$

	$\theta = \cos^{-1}\left(\frac{x}{2}\right)$ $\frac{d\theta}{dx} = \frac{-\frac{1}{2}}{\sqrt{1-\frac{x^2}{4}}}$ Sub $x = 1$ $\frac{d\theta}{dx} = \frac{-\frac{1}{2}}{\sqrt{1-\frac{1^2}{4}}}$ $\frac{d\theta}{dx} = -\frac{1}{\sqrt{3}}$ So, $\frac{dA}{dt} = 6 \times -\frac{1}{\sqrt{3}} \times \sqrt{3} = -6$ The shaded area, A, is decreasing at 6 units/time.	
Q14c(i)	$-\frac{1}{10}(a-2)(a-38) = -\frac{1}{10}(a^2 - 40a + 76)$ $-\frac{1}{10}(a-2)(a-38) = -\frac{a^2}{10} + 4a - 7.6$ $-\frac{1}{10}(a-2)(a-38) = 4a - \frac{a^2}{10} - 7.6$ $-\frac{1}{10}(a-2)(a-38) = \frac{1}{10}a(40-a) - 7.6$ $-\frac{1}{10}(a-2)(a-38) = -7.6 + \frac{1}{10}a(40-a)$	1 Mark Correct proof demonstrating all steps
Q14c(ii)	$\frac{da}{dt} = -7.6 + \frac{1}{10}a(40-a)$ $\frac{da}{dt} = -\frac{1}{10}(a-2)(a-38)$ $-\frac{1}{(a-2)(a-38)}da = \frac{1}{10}dt$ $\left(\frac{1}{36(a-2)} - \frac{1}{36(a-38)}\right)da = \frac{1}{10}dt$ $\int\left(\frac{1}{36(a-2)} - \frac{1}{36(a-38)}\right)da = \int \frac{1}{10}dt$ $\int \frac{1}{36(a-2)}da - \int \frac{1}{36(a-38)}da = \frac{1}{10}\int dt$ $\frac{1}{36}\log_e\|a-2\| - \frac{1}{36}\log_e\|a-38\| = \frac{t}{10} + C$	3 Marks Correct proof demonstrating all steps logically 2 Marks Demonstrates $\frac{1}{36}\log_e\|a-2\|$ $-\frac{1}{36}\log_e\|a-38\|$ $= \frac{t}{10} + C$ AND correct value of C

	$\frac{1}{36}\log_e\left\lvert\frac{a-2}{a-38}\right\rvert = \frac{t}{10} + C$ Sub $t = 0$ and $a = 1000$ $\frac{1}{36}\log_e\left\lvert\frac{1000-2}{1000-38}\right\rvert = \frac{0}{10} + C$ $\frac{1}{36}\log_e\left(\frac{998}{962}\right) = C$ $\frac{1}{36}\log_e\left(\frac{499}{481}\right) = C$ $\frac{1}{36}\log_e\left\lvert\frac{a-2}{a-38}\right\rvert = \frac{t}{10} + \frac{1}{36}\log_e\left(\frac{499}{481}\right)$ $\frac{1}{36}\log_e\left\lvert\frac{a-2}{a-38}\right\rvert - \frac{1}{36}\log_e\left(\frac{499}{481}\right) = \frac{t}{10}$ $\frac{1}{36}\log_e\left\lvert\frac{481(a-2)}{499(a-38)}\right\rvert = \frac{t}{10}$ $\frac{10}{36}\log_e\left\lvert\frac{481(a-2)}{499(a-38)}\right\rvert = t$ $t = \frac{5}{18}\log_e\left\lvert\frac{481(a-2)}{499(a-38)}\right\rvert$	$\frac{1}{36}\log_e\left(\frac{998}{962}\right) = C$ 1 Mark Demonstrates that $\frac{1}{36}\log_e\lvert a-2\rvert$ $-\frac{1}{36}\log_e\lvert a-38\rvert$ $= \frac{t}{10} + C$
Q14c(iii)	$t = \frac{5}{18}\log_e\left\lvert\frac{481(a-2)}{499(a-38)}\right\rvert$ Sub $a = 0$ $t = \frac{5}{18}\log_e\left\lvert\frac{481(0-2)}{499(0-38)}\right\rvert$ $t = 0.8281 \ldots$ years $t = 9.937 \ldots$ months $t \approx 10$ months It will take approximately 10 months.	1 Mark Correct answer